```
N = 9;
n = √N;
U = Table[u_i, {i, 0, N - 1}];
ArrayReshape[U, {n, n}] // MatrixForm
```

$$\begin{pmatrix} u_0 & u_1 & u_2 \\ u_3 & u_4 & u_5 \\ u_6 & u_7 & u_8 \end{pmatrix}$$

```
ArrayReshape[Table[g_i, {i, 0, N - 1}], {n, n}] // MatrixForm
```

$$\begin{pmatrix} g_0 & g_1 & g_2 \\ g_3 & g_4 & g_5 \\ g_6 & g_7 & g_8 \end{pmatrix}$$

1D

Basically, the matrix $A(\nabla A)$ is like A only instead of the pattern (1, –2, 1) the gradients are used directly to approximate the derivatives: (a, $-(a+b)$, b) with $a = g_{i-1} + g_i$ and $b = g_i + g_{i+1}$.

```
A_OneDim = Table[ { -g_i - g_(i+1)              i == 0 && j == 0
                  { -g_(i-1) - g_i              i == N - 1 && j == N - 1
                  { -g_(i-1) - 2 g_i - g_(i+1)  i == j
                  { g_(i-1) + g_i               j == i - 1
                  { g_i + g_(i+1)               j == i + 1
                  { 0                           True
                 , {i, 0, N - 1}, {j, 0, N - 1}];
```

```
A_OneDim // MatrixForm
```

$$\begin{pmatrix}
-g_0-g_1 & g_0+g_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
g_0+g_1 & -g_0-2g_1-g_2 & g_1+g_2 & 0 & 0 & 0 & 0 & 0 \\
0 & g_1+g_2 & -g_1-2g_2-g_3 & g_2+g_3 & 0 & 0 & 0 & 0 \\
0 & 0 & g_2+g_3 & -g_2-2g_3-g_4 & g_3+g_4 & 0 & 0 & 0 \\
0 & 0 & 0 & g_3+g_4 & -g_3-2g_4-g_5 & g_4+g_5 & 0 & 0 \\
0 & 0 & 0 & 0 & g_4+g_5 & -g_4-2g_5-g_6 & g_5+g_6 & 0 \\
0 & 0 & 0 & 0 & 0 & g_5+g_6 & -g_5-2g_6-g_7 & g_6+g_7 \\
0 & 0 & 0 & 0 & 0 & 0 & g_6+g_7 & -g_6-2g_7- \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & g_7+g_8
\end{pmatrix}$$

```
{A_OneDim . U}^T // MatrixForm
```

$$\begin{pmatrix}
(-g_0-g_1)\,u_0 + (g_0+g_1)\,u_1 \\
(g_0+g_1)\,u_0 + (-g_0-2g_1-g_2)\,u_1 + (g_1+g_2)\,u_2 \\
(g_1+g_2)\,u_1 + (-g_1-2g_2-g_3)\,u_2 + (g_2+g_3)\,u_3 \\
(g_2+g_3)\,u_2 + (-g_2-2g_3-g_4)\,u_3 + (g_3+g_4)\,u_4 \\
(g_3+g_4)\,u_3 + (-g_3-2g_4-g_5)\,u_4 + (g_4+g_5)\,u_5 \\
(g_4+g_5)\,u_4 + (-g_4-2g_5-g_6)\,u_5 + (g_5+g_6)\,u_6 \\
(g_5+g_6)\,u_5 + (-g_5-2g_6-g_7)\,u_6 + (g_6+g_7)\,u_7 \\
(g_6+g_7)\,u_6 + (-g_6-2g_7-g_8)\,u_7 + (g_7+g_8)\,u_8 \\
(g_7+g_8)\,u_7 + (-g_7-g_8)\,u_8
\end{pmatrix}$$

2D

Adding an additional dimension makes things a little bit more complicated because now more border cases need to be considered. The general idea is to consider the derivative in each dimension independently and add the results up (like the diffusion equation suggests).

```
A_x = Table[ { -g_(i-1) - g_i              i == j && Mod[i, n] == n - 1
             { -g_i - g_(i+1)              i == j && Mod[i, n] == 0
             { -g_(i-1) - 2 g_i - g_(i+1)  i == j
             { g_(i-1) + g_i               j == i - 1 && Mod[i, n] ≠ 0
             { g_i + g_(i+1)               j == i + 1 && Mod[i, n] ≠ n - 1
             { 0                           True
            , {i, 0, N - 1}, {j, 0, N - 1}];
```

```
A_x // MatrixForm
```

$$\begin{pmatrix}
-g_0-g_1 & g_0+g_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
g_0+g_1 & -g_0-2g_1-g_2 & g_1+g_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & g_1+g_2 & -g_1-g_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -g_3-g_4 & g_3+g_4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & g_3+g_4 & -g_3-2g_4-g_5 & g_4+g_5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & g_4+g_5 & -g_4-g_5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -g_6-g_7 & g_6+g_7 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & g_6+g_7 & -g_6-2g_7-g_8 & g_7+g_8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & g_7+g_8 & -g_7-g_8
\end{pmatrix}$$

$\{A_x.U\}^\intercal$ // MatrixForm

$$\begin{pmatrix}
(-g_0-g_1)\,u_0+(g_0+g_1)\,u_1 \\
(g_0+g_1)\,u_0+(-g_0-2g_1-g_2)\,u_1+(g_1+g_2)\,u_2 \\
(g_1+g_2)\,u_1+(-g_1-g_2)\,u_2 \\
(-g_3-g_4)\,u_3+(g_3+g_4)\,u_4 \\
(g_3+g_4)\,u_3+(-g_3-2g_4-g_5)\,u_4+(g_4+g_5)\,u_5 \\
(g_4+g_5)\,u_4+(-g_4-g_5)\,u_5 \\
(-g_6-g_7)\,u_6+(g_6+g_7)\,u_7 \\
(g_6+g_7)\,u_6+(-g_6-2g_7-g_8)\,u_7+(g_7+g_8)\,u_8 \\
(g_7+g_8)\,u_7+(-g_7-g_8)\,u_8
\end{pmatrix}$$

Expand$[(g_3+g_4)\,u_3+(-g_3-2g_4-g_5)\,u_4+(g_4+g_5)\,u_5]$ === Expand$[(g_3+g_4)\,(u_3-u_4)+(g_4+g_5)\,(u_5-u_4)]$

True

(* n-1 and N-n because of a n-by-n matrix/image, e. g. 2 and 6 with a 3-by-by matrix *)

$$A_y = \text{Table}\left[\begin{cases}
-g_j-g_{j+n} & i == j \,\&\&\, i \le n-1 \\
-g_{j-n}-g_j & i == j \,\&\&\, i \ge N-n \\
-g_{j-n}-2g_j-g_{j+n} & i == j \\
g_j+g_{j+n} & j == i-n \,\&\&\, i > 2 \\
g_{j-n}+g_j & j == i+n \,\&\&\, i < N-n \\
0 & \text{True}
\end{cases}, \{i, 0, N-1\}, \{j, 0, N-1\}\right];$$

A_y // MatrixForm

$$\begin{pmatrix}
-g_0-g_3 & 0 & 0 & g_0+g_3 & 0 & 0 & 0 & 0 & 0 \\
0 & -g_1-g_4 & 0 & 0 & g_1+g_4 & 0 & 0 & 0 & 0 \\
0 & 0 & -g_2-g_5 & 0 & 0 & g_2+g_5 & 0 & 0 & 0 \\
g_0+g_3 & 0 & 0 & -g_0-2g_3-g_6 & 0 & 0 & g_3+g_6 & 0 & 0 \\
0 & g_1+g_4 & 0 & 0 & -g_1-2g_4-g_7 & 0 & 0 & g_4+g_7 & 0 \\
0 & 0 & g_2+g_5 & 0 & 0 & -g_2-2g_5-g_8 & 0 & 0 & g_5+g_8 \\
0 & 0 & 0 & g_3+g_6 & 0 & 0 & -g_3-g_6 & 0 & 0 \\
0 & 0 & 0 & 0 & g_4+g_7 & 0 & 0 & -g_4-g_7 & 0 \\
0 & 0 & 0 & 0 & 0 & g_5+g_8 & 0 & 0 & -g_5-g_8
\end{pmatrix}$$

$\{A_y.U\}^\intercal$ // MatrixForm

$$\begin{pmatrix}
(-g_0-g_3)\,u_0+(g_0+g_3)\,u_3 \\
(-g_1-g_4)\,u_1+(g_1+g_4)\,u_4 \\
(-g_2-g_5)\,u_2+(g_2+g_5)\,u_5 \\
(g_0+g_3)\,u_0+(-g_0-2g_3-g_6)\,u_3+(g_3+g_6)\,u_6 \\
(g_1+g_4)\,u_1+(-g_1-2g_4-g_7)\,u_4+(g_4+g_7)\,u_7 \\
(g_2+g_5)\,u_2+(-g_2-2g_5-g_8)\,u_5+(g_5+g_8)\,u_8 \\
(g_3+g_6)\,u_3+(-g_3-g_6)\,u_6 \\
(g_4+g_7)\,u_4+(-g_4-g_7)\,u_7 \\
(g_5+g_8)\,u_5+(-g_5-g_8)\,u_8
\end{pmatrix}$$

Expand$[(g_1+g_4)\,u_1+(-g_1-2g_4-g_7)\,u_4+(g_4+g_7)\,u_7]$ === Expand$[(g_1+g_4)\,(u_1-u_4)+(g_4+g_7)\,(u_7-u_4)]$

True

(A_x+A_y) // MatrixForm

$$\begin{pmatrix} -2g_0-g_1-g_3 & g_0+g_1 & 0 & g_0+g_3 & 0 & 0 \\ g_0+g_1 & -g_0-3g_1-g_2-g_4 & g_1+g_2 & 0 & g_1+g_4 & 0 \\ 0 & g_1+g_2 & -g_1-2g_2-g_5 & 0 & 0 & g_2+g_5 \\ g_0+g_3 & 0 & 0 & -g_0-3g_3-g_4-g_6 & g_3+g_4 & 0 \\ 0 & g_1+g_4 & 0 & g_3+g_4 & -g_1-g_3-4g_4-g_5-g_7 & g_4+g_5 \\ 0 & 0 & g_2+g_5 & 0 & g_4+g_5 & -g_2-g_4-3g_5-g_8 \\ 0 & 0 & 0 & g_3+g_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & g_4+g_7 & 0 \\ 0 & 0 & 0 & 0 & 0 & g_5+g_8 \end{pmatrix}$$

```
erg = (Ax + Ay).U;
erg // MatrixForm
```

$$\begin{pmatrix} (-2g_0-g_1-g_3)u_0+(g_0+g_1)u_1+(g_0+g_3)u_3 \\ (g_0+g_1)u_0+(-g_0-3g_1-g_2-g_4)u_1+(g_1+g_2)u_2+(g_1+g_4)u_4 \\ (g_1+g_2)u_1+(-g_1-2g_2-g_5)u_2+(g_2+g_5)u_5 \\ (g_0+g_3)u_0+(-g_0-3g_3-g_4-g_6)u_3+(g_3+g_4)u_4+(g_3+g_6)u_6 \\ (g_1+g_4)u_1+(g_3+g_4)u_3+(-g_1-g_3-4g_4-g_5-g_7)u_4+(g_4+g_5)u_5+(g_4+g_7)u_7 \\ (g_2+g_5)u_2+(g_4+g_5)u_4+(-g_2-g_4-3g_5-g_8)u_5+(g_5+g_8)u_8 \\ (g_3+g_6)u_3+(-g_3-2g_6-g_7)u_6+(g_6+g_7)u_7 \\ (g_4+g_7)u_4+(g_6+g_7)u_6+(-g_4-g_6-3g_7-g_8)u_7+(g_7+g_8)u_8 \\ (g_5+g_8)u_5+(g_7+g_8)u_7+(-g_5-g_7-2g_8)u_8 \end{pmatrix}$$

```
Expand[erg[[5]]]
```

$g_1 u_1 + g_4 u_1 + g_3 u_3 + g_4 u_3 - g_1 u_4 - g_3 u_4 - 4 g_4 u_4 - g_5 u_4 - g_7 u_4 + g_4 u_5 + g_5 u_5 + g_4 u_7 + g_7 u_7$

```
Expand[erg[[5]]] ===
 Expand[(g1 + g4) u1 + (g3 + g4) u3 - (g1 + g3 + 4 g4 + g5 + g7) u4 + (g4 + g5) u5 + (g4 + g7) u7]
```

True

Data example

Small data example to check if the C++ implementation works as intended.

```
data = {{4.1, 1, 9}, {2, 10.9, 18}, {20.2, 8, 4.4}};
```

4.1	1	9
2	10.9	18
20.2	8	4.4

```
dataG = {{1, 0.054134817349126271, 1}, {0.044753942822362648, 0.083649808441938658, 0.81274382314694416}, {1, 0.99840255591054305, 1}};
```

1	0.054134817349126271	1
0.044753942822362648	0.083649808441938658	0.81274382314694416
1	0.99840255591054305	1

```
remapU = Table[u_((i-1)*3+j-1) → data[[i, j]], {i, 1, 3}, {j, 1, 3}] // Flatten
```

$\{u_0 \to 4.1, u_1 \to 1, u_2 \to 9, u_3 \to 2, u_4 \to 10.9, u_5 \to 18, u_6 \to 20.2, u_7 \to 8, u_8 \to 4.4\}$

```
remapG = Table[g_((i-1)*3+j-1) → dataG[[i, j]], {i, 1, 3}, {j, 1, 3}] // Flatten
```

$\{g_0 \to 1, g_1 \to 0.0541348, g_2 \to 1, g_3 \to 0.0447539,$
$g_4 \to 0.0836498, g_5 \to 0.812744, g_6 \to 1, g_7 \to 0.998403, g_8 \to 1\}$

```
FEDStep = ArrayReshape[
   Flatten[ArrayReshape[data, {1, 9}]] + 0.5 * 0.1 * erg /. Flatten[{remapU, remapG}], {3, 3}];
FEDStep // MatrixForm
```

3.82691	1.65325	9.39408
3.11756	10.936	15.6334
18.0302	9.01621	5.99238

Even at the corners the values are equal.

```
5.99237825980382`
5.9923782598038198
```